ON THE STRUCTURE OF SETS OF TRANSITIVE POINTS FOR CONTINUOUS MAPS OF THE INTERVAL

DAVID POKLUDA

Abstract. For continuous transitive maps of the unit interval we give the following characterization of transitive points, i.e. points with a dense trajectory. A set A is a set of transitive points of a continuous map f of the compact unit interval $I = [0,1]$ into itself if and only if the set $I \setminus A$ is a first category set of type F_σ and c-dense in I.

It is well-known that for a continuous map f of a compact metric space X, the set $\mathrm{Tr}(f)$ of transitive points of f, i.e. the set of points in X with dense trajectory, is a dense G_δ set. The argument is straightforward:
Denote by $\mathcal{B}$ a countable base of X. The set of transitive points of the map f is then of the form $\mathrm{Tr}(f) = \bigcap_{G \in \mathcal{B}} \bigcup_{n=1}^{\infty} f^{-n}(G)$ and therefore, a G_δ set, which is dense since any $\bigcup_{n=1}^{\infty} f^{-n}(G)$ is dense.

But, not every dense, G_δ set is a set of transitive points of a continuous map. In this paper we give a characterization of sets of transitive points (cf. Theorem 6 bellow). For the proof we use Proposition 3 and two Sharkovskii's results (Proposition 4 and 5). We will also state an alternate proof which is more elementary. This proof uses Lemma 7 and doesn't use the Sharkovskii's results.

The symbol I stands for the compact unit interval $[0,1]$ everywhere in this paper. Recall that for a map $f : I \to I$ and a point $x \in I$, the set $\{f^n(x), n \in \mathbb{N} \cup \{0\}\}$ is called *trajectory* of the point x for the map f. The set of limit points of the trajectory of the point x is called ω - *limit set.* The map f is called *transitive* if for every two nonempty open sets $A, B \subset I$ there is a positive number $n \in \mathbb{N}$ such, that $f^n(A) \cap B \neq \emptyset$. The map f is called *bitransitive* if the maps f, f^2 are transitive. The point $x \in I$ is called *transitive* point of the map f if the point x has a dense trajectory in I. The set of all transitive points of the map f is denoted by $\mathrm{Tr}(f)$. Finally the set is called *basic* if it is a maximal uncountable ω-limit set and the set contains a cycle. *Tent map* of the interval I is map $g : I \to I$, such that $g(x) = 2x$ for $x \in [0, \frac{1}{2}]$ and $g(x) = 2(1-x)$ for $x \in (\frac{1}{2}, 1]$. For more terminology see a standard book like [2].

Recall the following result (cf. [4], p. 355)

1991 *Mathematics Subject Classification.* Primary 58F03, 58F12, 26A18 .
This research was supported, in part, by the contract No. 201/97/0001 from the Grant Agency of the Czech Republic. Support of this institution is gratefully acknowledged.

Proposition 1. *(Alexandroff-Hausdorff) An uncountable Borel set A of a completely separable metric space contains a set that is homeomorphic to the Cantor set.*

Lemma 2. *Let D be an F_σ, first category and c-dense set. Then there is a system of perfect and nowhere dense sets $\{H_n\}_{n=1}^\infty$ such that $H_n \subset H_{n+1}$ for each n and any contiguous interval J to a set H_n, $J \cap H_{n+1} \neq \emptyset$, $\inf J \cap H_{n+1} = \min J$, $\sup J \cap H_{n+1} = \max J$ and $\bigcup H_n = D$.*

Proof. Put $D = \bigcup_{n=1}^\infty F_n$, where F_n are nowhere dense closed sets, and denote $F = F_1$. Let $\{U_n\}_{n=1}^\infty$ be the system of intervals contiguous to F. For every n find a Cantor set $C_n \subset \overline{U_n} \cap D$ with $\inf C_n = \min \overline{U_n}$ and $\sup C_n = \max \overline{U_n}$ (cf. Proposition 1). Put $H_1 = F \cup \bigcup_{n=1}^\infty C_n$ and note that H_1 is a Cantor set.

Suppose by induction that $H_1, \dots, H_n$ are defined. Denote $F = F_{n+1} \cup H_n$ and $\{U_n\}_{n=1}^\infty$ the system of intervals contiguous to F. For every n find a Cantor set $C_n \subset \overline{U_n} \cap D$ such that $\inf C_n = \min \overline{U_n}$ and $\sup C_n = \max \overline{U_n}$ (cf. Proposition 1). To finish the proof denote $H_{n+1} = F \cup \bigcup_{n=1}^\infty C_n$, it is a Cantor set. □

The following lemma is used in the prove of the main theorem, and it is interesting even by itself. Even though the prove can be found in [3] we will prove it here since our argument seems to be more simple.

Proposition 3. *Suppose A, B are first category, F_σ and c-dense sets in $I = [0,1]$. There is a homeomorphism $\varphi : I \to I$ where $\varphi(A) = B$.*

Proof. By Lemma 2, $A = \bigcup_{n=1}^\infty A_n$ and $B = \bigcup_{n=1}^\infty B_n$ where A_n are perfect, nowhere dense, $A_n \subset A_{n+1}$ for every n and similarly for B_n. To show that the sets A and B are homeomorphic, define a system of maps $\{\varphi_n\}_{n=1}^\infty$, $\varphi_n : I \to I$, for every n where $\varphi_n(A_i) = B_i, i \leq n$ and $\varphi_k|_{A_k} = \varphi_{n+k}|_{A_k}, n \geq 0$.

Denote by φ_1 a homeomorphism that maps A_1 onto B_1. Suppose by induction that $\varphi_1, \dots, \varphi_n$ are already defined. Let J be an interval contiguous to A_n (clearly the interval $\varphi_n(J)$ is an interval contiguous to B_n, because the map φ_n was defined this way). The symbol φ_{n+1} will now denote a homeomorphic map where $\varphi_{n+1}|_J : J \to \varphi_n(J)$ and $\varphi_{n+1}(J \cap A_{n+1}) = \varphi_n(J) \cap B_{n+1}$. To finish the proof note that $\|\varphi_{n+k} - \varphi_n\| \leq \sup_P \|P\|$ for every n, where P are contiguous intervals to B_n. Denote $\varphi = \lim_{n\to\infty} \varphi_n$. This map φ satisfies the conditions for the homeomorphic map between the sets A and B. □

Proposition 4. *(A. N. Sharkovskii) Suppose $\tilde{\omega}$ is basic set and $\omega_f(x) \subset \tilde{\omega}$. Then the points $y \in \tilde{\omega}$ for which $\omega_f(y) = \omega_f(x)$ are everywhere dense in $\tilde{\omega}$.*

Proposition 5. *(A. N. Sharkovskii, cf.* [5]*) Assume that $f \in \mathcal{C}(I, I)$. If f has a periodic orbit of period different from a power of two then there exists an uncountable set $B \subset I$ such that $\{\omega_f(x)\}_{x\in B}$ is ordered by inclusion.*

Let us note that in [6] it is stated without proof that the two conditions in Proposition 5 are equivalent. However this is not true, cf. Theorem B in [1].

Theorem 6. *A set $T \subset I$ is a set of transitive points for a continuous map $f : I \to I$ if and only if $I \setminus T$ is a first category set, F_σ and c-dense in I.*

Proof. Let f be a transitive map. Clearly $\mathrm{Tr}(f) = \{x; \omega_f(x) = I\}$ is G_δ and dense set. The set I is therefore a basic set of the map f. It is sufficient to show that the set $I \setminus \mathrm{Tr}(f)$ is c-dense. Since $I = \tilde{\omega}$ is basic and there is $x \in I$ such that $\omega_f(x) \subset \tilde{\omega}$ and the set $\{y \in I; \omega_f(y) = \omega_f(x)\}$ is dense in I (cf. Proposition 4). Moreover the set $I = \tilde{\omega}$ is basic and so there is a decreasing sequence of sets (with respect to inclusions) of ω-limit sets (cf. Proposition 5).

To prove the converse denote by g the tent map of the interval I. The set $\mathrm{Tr}(g)$, the set of transitive points of the map g, satisfies the conditions for a transitive set. Suppose a set T is G_δ and $I \setminus T$ is c-dense. It is sufficient to show that the set is homeomorphic with the set $\mathrm{Tr}(g)$. Denote $\varphi : T \to \mathrm{Tr}(g)$ be a homeomorphic map (cf. Proposition 3). The map $f = \varphi^{-1} \cdot g \cdot \varphi$ is our continuous map that holds $\mathrm{Tr}(f) = T$. □

The above proof of the theorem is not trivial because of the two Sharkovskii's results. We can prove this lemma in a more elementary way by using the following lemma.

Lemma 7. *Let f be a transitive continuous map $f : I \to I$. Then for every interval J there is an invariant nowhere dense set F_J such that $\#(J \cap F_J) = c$.*

Proof. Assume, we are given an interval J. If the map f is bitransitive we can find two disjoint intervals J_0, J_1, where $J_0 \cup J_1 \subset J$ and $n \in \mathbb{N}$ such that $f^n(J_0) \cap f^n(J_1) \supset J_0 \cup J_1$. If the map is not bitransitive then there are two sets A, B such that $f(A) = B, f(B) = A$ and $f^2|_A, f^2|_B$ are bitransitive. In this case we will choose the sets J_0 and J_1 from $J \cap A$ where $J \cap A \neq \emptyset$ (if $J \cap A = \emptyset$ then we will choose the sets from $J \cap B$). Denote f^n by g and set $G = \bigcap_{k=0}^{\infty} g^{-k}(J_0 \cup J_1)$. Then the set G is uncountable and invariant, since $g(G) = g(\bigcap_{k=0}^{\infty} g^{-k}(J_0 \cup J_1)) \subset \bigcap_{k=0}^{\infty} g^{-k+1}(J_0 \cup J_1) \subset G$. Note that for every $\alpha \in \{0, 1\}^{\mathbb{N}}$ there is $x_\alpha \in G$ such that $g^n(x_\alpha) \in J_{\alpha_n}$.To finish the proof put $F_J = \bigcup_{k=0}^{n-1} f^k(G)$. □

We will now state the proof of the main theorem based on the previous lemma.

Proof. (Proof of the Theorem 6.) Let f be a transitive map. Clearly $Tr(f) = \{x; \omega_f(x) = [0, 1]\}$ is G_δ and dense set. Because of the previous lemma we see that the set $I \setminus Tr(f)$ is c-dense. The prove of the converse is the same as in the previous proof of the main theorem. □

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Institute of Mathematics, Silesian University, Opava, CZ 74601, Czech Republic
E-mail address: David.Pokluda@math.slu.cz