ON THE TRANSITIVE AND ω-LIMIT POINTS OF THE CONTINUOUS MAPPINGS OF THE CIRCLE

DAVID POKLUDA

ABSTRACT. We extend the recent results from the class $\mathcal{C}(I, I)$ of continuous maps of the interval to the class $\mathcal{C}(\mathbb{S}, \mathbb{S})$ of continuous maps of the circle. Among others, we give a characterization of ω-limit sets and give a characterization of sets of transitive points for these maps.

1. INTRODUCTION

Continuous maps of the interval and continuous maps of the circle have many properties in common. However to transfer results from the interval to the circle, we usually have to make some modifications that do not have to be seen at once. A good example is the classical result of the continuous maps of the interval, the Sharkovsky's theorem. The theorem does not hold for the maps of the circle in this classical version and it had to be modified for this class of maps (see [3]). In this paper we extend results from [1], [6] and [7] to the class of continuous maps of the circle.

In the sequel, X denotes either the compact unit interval or the unit circle $\mathbb{S}$. Main theorems of the paper are the following.

Theorem 1.1. *A non empty compact set $W \subset X$ is an ω-limit set of a map $f \in \mathcal{C}(X, X)$ if and only if W is either a finite collection of compact intervals or a nowhere dense set.*

Theorem 1.2. *There is a map $g \in \mathcal{C}(X, X)$ such that for any $f \in \mathcal{C}(X, X)$ and any ω-limit set $\omega_f(x) \neq X$, there is a homeomorphism φ from X into X and a point $y \in X$ such that $\varphi(\omega_f(x)) = \omega_g(y)$.*

Theorem 1.3. *A set $T \subset X$ is a set of transitive points for a map $f \in \mathcal{C}(X, X)$ if and only if $X \setminus T$ is a first category, F_σ set c-dense in X or $T = X$ and $X = \mathbb{S}$.*

Throughout the paper the set of continuous functions from a compact metric space Y into itself will be denoted by $\mathcal{C}(Y, Y)$. Symbols I and $\mathbb{S}$ denote the unit interval $[0, 1]$ and the circle $\{z \in \mathbb{C}; |z| = 1\}$, respectively. Denote by $e : \mathbb{R} \to \mathbb{S}$ the natural projection defined by $e(x) = \exp(2\pi i x)$. Recall that the *trajectory* of a point x under a map f is the sequence $\{f^n(x)\}_{n=0}^{\infty}$, where f^n is the n-th

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iterate of f. If there is $k \geq 1$ such that $f^k(x) = x$ and $f^n(x) \neq x$ for every $n = 1, \dots, k-1$, then x is a *periodic* point with the *period* k. We denote by $P(f)$ the set of periodic points of f. The set of limit points of the trajectory of x is called *ω-limit set* and we denote the set by $\omega_f(x)$. The map f is *transitive* if for every two nonempty open sets $A, B \subset X$ there is a positive number $n \in \mathbb{N}$ such, that $f^n(A) \cap B \neq \emptyset$, or equivalently, if there is a point $x \in X$ whose trajectory is dense in X; any such x is called a *transitive* point of f. The set of transitive points of f is denoted by $\mathrm{Tr}(f)$. Maps $f : Y_1 \to Y_1$ and $g : Y_2 \to Y_2$ are *topologically conjugate* if there exists a homeomorphism $h : Y_1 \to Y_2$ such that $h \circ f(x) = g \circ h(x)$ for $x \in Y_1$. For more terminology see a standard book like [3].

2. Proof of the main theorems

Note that the map $\tilde{e} : (v, v+1) \to \mathbb{S} \setminus \{e(v)\}$ obtained by restricting e to the interval $(v, v+1)$, is a homeomorphism. It is clear that if we define a map $h(x) = e(x+b)$ where $b \in \mathbb{R}$ then $\tilde{h} : (0,1) \to \mathbb{S} \setminus \{e(b)\}$ is a homeomorphism (see Lemma 3.1.3 in [2]). The following lemma is obvious.

Lemma 2.1. *Let $A \subset \mathbb{S}$ be closed non-degenerated interval and $A \neq \mathbb{S}$. There is a homeomorphism φ from A onto I.*

Proof of the Theorem 1.1. The proof for $X = I$ is known and can be found in [1], for example. It remains to consider the case when $X = \mathbb{S}$. Let $W \subset \mathbb{S}$ be a compact set which is either nowhere dense or a finite collection of compact intervals. In the case $W = \mathbb{S}$ the corresponding continuous map f is an irrational rotation of the circle. Otherwise there is a maximal open interval $U \subset \mathbb{S} \setminus W$. Denote by A the set $\mathbb{S} \setminus U$. Apply Lemma 2.1 to obtain a map $\varphi : A \to [0,1]$. From the characterization of ω-limit sets of $\mathcal{C}(I, I)$ there is a map $h : I \to I$ and a point $x \in I$ such that $\varphi(W) = \omega_h(x)$. Define $f|_A = \varphi^{-1} \circ h \circ \varphi$ and let $f|_U$ be a linear extension so that f is continuous on $\mathbb{S}$. Then $W = \omega_f(\varphi^{-1}(x))$.
Conversely, let $W = \omega_f(x)$, for an $f \in \mathcal{C}(\mathbb{S}, \mathbb{S})$. First assume that W contains an interval. Let J_1 be a maximal closed subinterval of W. Since J_1 contains more than one point of the trajectory of the point x, we have $f^i(J_1) \cap J_1 \neq \emptyset$ for some $i > 0$. It follows from Lemma 4.1.1 in [3] that there exists an integer $p > 0$ such that the closed connected sets $J_k = f^{k-1}(J_1)$ $(1 \leq k \leq p)$ are pairwise disjoint and $f(J_p) \subset J_1$. Moreover $W = \bigcup_{k=1}^{p} J_k$, since if $f^m(x) \in J_1$ then $f^n(x) \in \bigcup_{k=1}^{p} J_k$ for every $n > m$. Since W is strongly invariant, it follows that $f(J_p) = J_1$ and each J_k is an interval. It is clear that the set is compact and nowhere-dense otherwise. □

Proof of the Theorem 1.2. In [6] we proved the theorem for $X = I$; denote by h the corresponding "universal" function from I into I. It remains to consider the case when $X = \mathbb{S}$. It is enough to define a function ψ by shrinking the function h from $[0,1]$ to $[1/3, 2/3]$ and extended it linearly to the whole circle so that ψ is continuous and $\psi(0) = 0$ and $\psi(1) = 1$. To conclude denote $g = e \circ \psi \circ (e|_{[0,1)})^{-1}$. □

We cannot improve the theorem by extending the function to cover even the set $F = X$. The case $F = \mathbb{S}$ is not possible, because the only homeomorphic copy F

and a function $g \in \mathcal{C}(\mathbb{S},\mathbb{S})$ possessing this ω-limit set cannot have any other ω-limit set.

Let us recall an auxiliary proposition from [7] and a Blokh's result from [4], respectively.

Proposition 2.2. *Suppose A, B are first category, F_σ and c-dense sets in $I = [0,1]$. Then there is a homeomorphism $\varphi : I \to I$ with $\varphi(A) = B$.*

Proposition 2.3. *Suppose $f \in \mathcal{C}(\mathbb{S},\mathbb{S})$ is a transitive map. Then there is a positive integer m, such that $\mathbb{S} = \bigcup_{i=0}^{m-1} K_i$, where all the sets K_i are connected compact sets, $K_i \cap K_j$ is finite for $i \neq j$, $f(K_i) = K_{(i+1) \bmod m}$ and two cases are possible:*
(1) $P(f) \neq \emptyset$, then $f^{mq}|_{K_i}$ is transitive, where $i = 0, 1, \ldots, m-1$ and q is an arbitrary positive integer,
(2) $P(f) = \emptyset$, then $m = 1$, $K_0 = \mathbb{S}$ and f is conjugate to an irrational rotation.

Lemma 2.4. *Let f be a transitive continuous map $f : \mathbb{S} \to \mathbb{S}$ which is not topologically conjugate to an irrational rotation. Then for every interval J there is an invariant nowhere dense set F_J such that $\#(J \cap F_J) = c$.*

Proof. Let $J \subset \mathbb{S}$. First we show that there are disjoint compact intervals $J_0, J_1 \subset J$, and a positive integer n such that

$$f^n(J_0) \cap f^n(J_1) \supset J_0 \cup J_1. \tag{1}$$

Then the construction of F_J is simple. Put $F = \bigcap_{k=0}^{\infty} f^{-nk}(J_0 \cup J_1)$. Then $f^n(F) \subset F$ and $\#F = c$ since for any sequence $\{\alpha_k\}_{k=1}^{\infty}$ of zeros and ones, there is a point $x \in F$ such that $f^{kn}(x) \in J_{\alpha_k}$, for any k. Put $F_J = F \cup f(F) \cup \ldots \cup f^{n-1}(F)$.

It remains to prove (1). Let m and $K_0, \ldots, K_{m-1}$ be as in Proposition 2.3. We may assume $J \subset K_0$. Let J_0 be a compact interval contained in the interior $\mathrm{Int}(J)$ of J. Since f is not conjugate to an irrational rotation the periodic points are dense in $\mathbb{S}$ (cf., e.g., Theorems 7.3 and 7.2 in [5]). Hence there is a periodic point $a_0 \in J_0$ of period n_0. Since f^{mq} is transitive on K_0 for all q, the system $\{f^{kn_0m}(J_0)\}_{k=0}^{\infty}$ is a nest and its union A either contains $\mathrm{Int}(K_0)$, or $A \supset \mathrm{Int}(K_0) \setminus \{t\}$, for some t (the latter case may appear only when $m = 1$, and hence, $K_0 = \mathbb{S}$).

Put $\tilde{J} = \mathrm{Int}(J) \cap A$, and let $J_1 \subset \tilde{J} \setminus J_0$ be a compact interval. Then $f^{kn_0m}(J_0) \supset J_0 \cup J_1$ whenever $k \geq k_0$. Let $a_1 \in J_1$, be a periodic point of period n_1. Then similarly, $f^{kn_0n_1m}(J_1) \supset J_0 \cup J_1$ if $k \geq k_1$. Put $n = k_0k_1n_0n_1m$. This yields (1). $\square$

We will continue by proving the Theorem 1.3 about structure of the sets of transitive points.

Proof of the Theorem 1.3. In [7] we proved the theorem for $X = I$. It remains to consider the case when $X = \mathbb{S}$. If f is conjugate to an irrational rotation then $\mathrm{Tr}(f) = \mathbb{S}$. So let f be a transitive map that is not conjugate to an irrational rotation. Clearly $Tr(f) = \{x; \omega_f(x) = \mathbb{S}\}$ is G_δ and dense set. The argument is straightforward: If $\mathcal{B}$ is a countable base of $\mathbb{S}$, then $\mathrm{Tr}(f) = \bigcap_{G \in \mathcal{B}} \bigcup_{n=1}^{\infty} f^{-n}(G)$ and therefore, $\mathrm{Tr}(f)$ is a G_δ set, which is dense since any $\bigcup_{n=1}^{\infty} f^{-n}(G)$ is dense. Because of the previous lemma we see that the set $\mathbb{S} \setminus Tr(f)$ is c-dense.
To prove the converse suppose $T \subseteq \mathbb{S}$. If $T = \mathbb{S}$ take for f an irrational rotation.

So let $\mathbb{S} \setminus T$ be a first category set, F_σ and c-dense in $\mathbb{S}$. There is a $b \in \mathbb{S}$ such that $b \notin T$. Denote $a = e^{-1}(b)$ and $\tilde{e} = e|_{(a,a+1)}$. Because the map $\tilde{e}^{-1}$ is homeomorphism from $\mathbb{S} \setminus \{b\}$ onto the interval $(a, a+1)$, the set $(a, a+1) \setminus \tilde{T}$ is a first category set, F_σ and c-dense, where $\tilde{T} = \tilde{e}^{-1}(T)$. Let g be a transitive map of the interval $J = [a, a+1]$ with $g(a) = g(a+1)$. From the case when $X = J$ we get that the set $J \setminus \mathrm{Tr}(g)$ is a first category set, F_σ and c-dense in J. According to the Proposition 2.2 there is a homeomorphism $\varphi : J \to J$ such that $\varphi(\tilde{T}) = \mathrm{Tr}(g)$. Denote $f|_{\mathbb{S}\setminus\{b\}} = \tilde{e} \circ \varphi^{-1} \circ g \circ \varphi \circ \tilde{e}^{-1}$. Since $g(a) = g(a+1)$, we have $\lim_{x\to b^-} f(x) = \lim_{x \to b^+} f(x)$ and hence the map f can be continuously extended to the whole circle $\mathbb{S}$. $\square$

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Mathematical Institute, Silesian University, Opava, CZ 74601, Czech Republic
E-mail address: David.Pokluda@math.slu.cz