CHARACTERIZATION OF ω-LIMIT SETS OF CONTINUOUS MAPS OF THE CIRCLE

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Abstract. In this paper we extend results of Blokh, Bruckner, Humke and Smítal [Tran. Amer. Math. Soc. **348** (1996), 1357–1372] about characterization of ω-limit sets from the class $\mathcal{C}(I,I)$ of continuous maps of the interval to the class $\mathcal{C}(\mathbb{S},\mathbb{S})$ of continuous maps of the circle. Among others we give geometric characterization of ω-limit sets and then we prove that the family of ω-limit sets is closed with respect to the Hausdorff metric.

1. Introduction

Continuous maps of the interval and continuous maps of the circle have many properties in common. Some of them are proved in [6]. In this paper we extend results proved in [3] from the class $\mathcal{C}(I,I)$ of continuous maps of the interval to the class $\mathcal{C}(\mathbb{S},\mathbb{S})$ of continuous maps of the circle by using the same technique used in [6]. Other results concerning continuous maps of the circle can be found in [1] or [5].

Throughout the paper, the symbols I and $\mathbb{S}$ denote the unit interval $[0,1]$ and the circle $\{z \in \mathbb{C}; |z| = 1\}$, respectively, and X denotes either I or $\mathbb{S}$. Denote by $\mathbb{S}_b$ the circle cut at a point $b \in \mathbb{S}$, i.e. $\mathbb{S}_b = \mathbb{S} \setminus \{b\}$. Let $e : \mathbb{R} \to \mathbb{S}$ be the natural projection defined by $e(x) = \exp(2\pi i x)$. Note that the map $\tilde{e} : (v, v+1) \to \mathbb{S}_{e(v)}$ obtained by restricting e to the interval $(v, v+1)$, is a homeomorphism. It is clear that if we define a map $h_v(x) := e(x+v)$, where $v \in \mathbb{R}$, then $\tilde{h}_v := h_v|_{(0,1)}$ is a homeomorphism from $(0,1)$ onto $\mathbb{S} \setminus \{e(v)\}$ (see Lemma 3.1.3 in [1]). We say that $\tilde{h}_v(x) \le \tilde{h}_v(y)$ whenever $x \le y$. For an interval $A \subset \mathbb{S}_{e(v)}$ a point a is called the *left endpoint*, resp. the *right endpoint*, of A if $a \le x$, resp. $x \le a$, for every $x \in A$. Recall that the *trajectory* of a

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point x under a map f is the sequence $\{f^n(x)\}_{n=0}^{\infty}$, where f^n is the n-th iteration of f. The set of limit points of the trajectory of x is called *ω-limit set* and we denote the set by $\omega_f(x)$. A set $\{U_0, \dots, U_{n-1}\}$ of mutually disjoint intervals is called a *cycle of intervals* if $f(U_i) = U_{i+1}$ for $i = 0, 1, \dots, n-2$ and $f(U_{n-1}) = U_0$. The map f is *transitive* if for every two non-empty open sets V, W there is a positive integer n such, that $f^n(V) \cap W \neq \emptyset$. Two maps $f : Y_1 \to Y_1$ and $g : Y_2 \to Y_2$ are *topologically conjugate* if there exists a homeomorphism $\varphi : Y_1 \to Y_2$ such that $\varphi \circ f(x) = g \circ \varphi(x)$ for any $x \in Y_1$. For more terminology see standard books like [1] or [2].

Now we introduce some notions used in [3] and modified for maps from $\mathcal{C}(\mathbb{S}, \mathbb{S})$. We say that a set $A \subset \mathbb{S}$ is *T-side* or *T-unilateral neighborhood* (T means either "left" or "right") of an $x \in \mathbb{S}$ if the set A is a closed interval and the point x is T endpoint of the set A.
Let $U = U_0 \cup \ldots \cup U_{N-1}$ be a union of pairwise disjoint non-degenerate closed intervals and $f \in \mathcal{C}(\mathbb{S}, \mathbb{S})$. For any set $K \subset U$ let $f_U(K) = f(K) \cap U$ (this may be empty). Inductively define $f_U^n(K) = f_U(f_U^{n-1}(K))$. Define $\tilde{K} \equiv \tilde{K}(U) = \bigcup_{i=1}^{\infty} f_U^i(K)$; although $\tilde{K}$ depends on U, to avoid convoluted notation we use $\tilde{K}$ whenever the set U is evident. Let $A \subset \mathbb{S}$ be a closed set and $x \in A$. We say that a side T of a point x is *A-covering* if for any union of finitely many closed intervals U such that $A \subset \mathrm{Int}(U)$ and any closed T-unilateral neighborhood $V(x)$ there are finitely many components of $\tilde{V}(x)$ such that the closure of their union covers A. If T is an A-covering side of x then any T-unilateral neighborhood $V(x)$ is also said to be *A-covering.* We call the set A *locally expanding* according to the map f if every $x \in A$ has an A-covering side.

The main theorems of this paper are the following.

Theorem 1.1. *Let f be a map in $\mathcal{C}(X, X)$. A closed set $A \subset X$ is an ω-limit set if and only if it is locally expanding.*

Theorem 1.2. *Let $\{\omega_n\}_{n=1}^{\infty} = \{\omega_f(x_n)\}_{n=1}^{\infty}$ be a sequence of ω-limit sets of a continuous map $f \in \mathcal{C}(X, X)$ and let a point a have a side T, such that for any T-unilateral neighborhood V of a, there exists a positive integer N such that for each $n \geq N$, the orbit of x_n enters V infinitely many times. Then $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n}$ is an ω-limit set.*

Theorem 1.3. *Let f be a map in $\mathcal{C}(X, X)$. Then the family of all ω-limit sets of f endowed with the Hausdorff metric is compact.*

2. Proof of the main theorems

Let $b \in \mathbb{S}$ and $f \in \mathcal{C}(\mathbb{S}, \mathbb{S})$. We denote by $e^{-1}(b)$ the point $x \in [0,1)$ such that $e(x) = b$. In the rest of the paper by h we denote the map $\tilde{h}_{e^{-1}(b)}$ whenever the point $b \in \mathbb{S}$ is evident and by A^* we mean the preimage of the set $A \subset \mathbb{S}_b$ under the map $\tilde{h}_{e^{-1}(b)}$. Denote by S the set $\mathbb{S} \setminus \bigcup_{n=0}^{\infty} f^{-n}(b)$. Now we can define a map $f^* \in \mathcal{C}(S^*, S^*)$ as

$$f^* := h^{-1} \circ f \circ h|_{S^*}.$$

The map f^* is defined only on the subset S^* of the interval $(0,1)$, but we overcome this difficulty using Lemma 2.1.

Lemma 2.1. *Let $f \in \mathcal{C}(X, X)$ and $A \subset X$ be a locally expanding set according to the map f. Then the set A is invariant, i.e. $f(A) \subset A$.*

Proof. In the case when $X = I$ the lemma is proved in [3] (Lemma 2.5). It remains to consider the case $X = \mathbb{S}$. The case $A = \mathbb{S}$ is trivial. Let $A \subset \mathbb{S}_b$, $x \in A$ and $f(x) \notin A$. Then there exists a union of finitely many intervals $U = U_0 \cup \ldots \cup U_{n-1}$, $U \supset A$ such that for any sufficiently small neighborhood V of x we have $f(V) \cap U = \emptyset$. The definition of $\tilde{V}$ implies that $\tilde{V} = \emptyset$ which is a contradiction. $\square$

Lemma 2.2. *A set $A \subset \mathbb{S}$ is a T-side of a point $x \in \mathbb{S}$ if and only if the set A^* is a T-side of the point x^*.*

The proof is omitted.

Lemma 2.3. *If the whole circle $\mathbb{S}$ is locally expanding with respect to a map $f \in \mathcal{C}(\mathbb{S}, \mathbb{S})$ then f is transitive.*

Proof. Take two nonempty open sets V, W. Since a point $x \in \mathrm{Int}(V)$ has $\mathbb{S}$ covering side then $\tilde{V} = \mathbb{S}$ and hence there is a positive integer n such that $f^n(V) \cap W \neq \emptyset$. This proves that the map f is transitive. $\square$

Lemma 2.4. *Let f be a map in $\mathcal{C}(\mathbb{S}, \mathbb{S})$. A closed set $A \subset \mathbb{S}_b$ is locally expanding according to the map f if and only if the set $A^* \subset (0,1)$ is locally expanding according to the map f^*.*

Proof. First assume that the set A^* is locally expanding. Hence the sets A^*, A are closed and by Lemma 2.1 the set A^* is invariant and $A^* \subset S^*$. Take a point $x \in A$. Since the set A^* is locally expanding the point x^* has an A^*-covering side T^*. By Lemma 2.2 the set T is a side of the point x. Take a union of finitely many closed intervals $U \subset \mathbb{S}_b$ such that $A \subset \mathrm{Int}(U)$ and any closed T-unilateral neighborhood $V(x)$. Using the assumptions there are finitely many components of $\tilde{W}$ where

$W = V(x)^*$ such that the closure of their union covers A^* and clearly $\tilde{W} \subset (0,1)$. Hence the set $\tilde{V}(x)$ has finitely many components such that the closure of their union covers A as well. Thus the set A is locally expanding.

The proof of the converse is analogous. □

Lemma 2.5. *A set $A \subset \mathbb{S}_b$ is an ω-limit set of the map $f \in \mathcal{C}(\mathbb{S},\mathbb{S})$ if and only if the set A^* is an ω-limit set of the map f^*.*

Proof. First consider the closed set $A \subset \mathbb{S}_b$ to be an ω-limit set. There is a point $x_0 \in \mathbb{S}$ such that $\omega_f(x_0) = A$. If there are two positive integers $m_1 < m_2$ such that $f^{m_1}(x_0) = f^{m_2}(x_0) = b$ then the ω-limit set A is finite and $b \in A$ which is a contradiction. We may assume that $f^n(x_0) \neq b$ for every positive integer n (in the case when there is just one positive integer m such that $f^m(x_0) = b$ we replace x_0 by $f^{m+1}(x_0)$) and thus $\{f^n(x_0)\}_{n=0}^{\infty} \subset S$. Hence $(\{f^n(x_0)\}_{n=0}^{\infty})^* \subset S^*$ and we have $\omega_{f^*}(x_0^*) = (\omega_f(x_0))^* = A^*$.

The proof of the converse is analogous. □

Before stating the next lemma, let us recall one of Blokh's results from [4].

Proposition 2.6. *Suppose that $f \in \mathcal{C}(\mathbb{S},\mathbb{S})$ is a transitive map. Then there is a positive integer m, such that $\mathbb{S} = \bigcup_{i=0}^{m-1} K_i$, where all the K_i are connected compact sets, $K_i \cap K_j$ is finite for $i \neq j$, $f(K_i) = K_{i+1}$, $i = 0,1,\dots,m-2$, $f(K_{m-1}) = K_0$ and two cases are possible:*
(1) $P(f) \neq \emptyset$; then $f^{mq}|_{K_i}$ is transitive for any $i = 0,1,\dots,m-1$ and any positive integer q,
(2) $P(f) = \emptyset$; then $m = 1$, $K_0 = \mathbb{S}$ and f is conjugate to an irrational rotation.

Lemma 2.7 (Lemma 2.6 in [3] for $\mathcal{C}(I,I)$)**.** *Let f be a map in $\mathcal{C}(\mathbb{S},\mathbb{S})$ and $A \subset \mathbb{S}$ be a locally expanding set according to the map f with non-empty interior. Then A is a cycle of intervals and $f|_A$ is transitive.*

Proof. Suppose that $A = \mathbb{S}$. By Lemma 2.3 the map f is transitive and by Proposition 2.6 the set A must be a cycle of intervals. Suppose that $A \subset \mathbb{S}_b$. Since A is locally expanding then by Lemma 2.1 $A \subset S$ and by Lemma 2.4 the set $A^* \subset S^*$ is locally expanding. By Lemma 2.6 in [3] the set A^* is a cycle of intervals $A_0^*,\dots,A_{n-1}^*$ and $f^*|_{A^*}$ is transitive. The map h is a homeomorphism and hence the set $A = h(A^*) = h(A_0^*) \cup \ldots \cup h(A_{n-1}^*)$ and

$$f(A_i) = \left(h \circ f^* \circ h^{-1}|_S\right)(A_i) = \left(h \circ f^* \circ h^{-1}|_S\right)(h(A_i^*)) = h(f^*(A_i^*)) = h(A_{i+1}^*) = A_{i+1},$$

where $A_j = h(A_j^*)$ and j is taken modulo n. This means that A is a cycle of intervals. It remains to show that $f|_A$ is transitive when $A \subset \mathbb{S}_b$. Take two open sets $V, W \subset A$. Then the sets $V^*, W^* \subset S^*$ are open sets and so there is a positive integer n such that $(f^*)^n(V^*) \cap W^* \neq \emptyset$. Hence

$$f^n(V) \cap W = (h \circ (f^*)^n \circ h^{-1}|_S)(V) \cap W = h((f^*)^n(V^*) \cap W^*) \neq \emptyset.$$

□

Lemma 2.8 (Lemma 2.7 in [3] for $\mathcal{C}(I, I)$)**.** *Let f be a map in $\mathcal{C}(\mathbb{S}, \mathbb{S})$ and $A \subset \mathbb{S}$ be a locally expanding or an ω-limit set. Then $f(A) = A$.*

Proof. The case of an ω-limit set is trivial and well known. Let A be a locally expanding set. When $A = \mathbb{S}$ then f is transitive (Lemma 2.3) and the lemma is proved. It remains to consider the case when $A \subset \mathbb{S}_b$. By Lemma 2.1 $A \subset S$, and by Lemma 2.7 in [3] we have $f^*(A^*) = A^*$. Clearly

$$f(A) = \left(h \circ f^* \circ h^{-1}|_S\right)(A) = h(f^*(A^*)) = h(A^*) = A.$$

□

We continue by proving the main theorems.

Proof of Theorem 1.1. In the case when $X = I$ the theorem is proved in [3] (Theorem 2.12). It remains to consider the case when $X = \mathbb{S}$. First we show that if $A = \omega_f(x)$ is an ω-limit set then A is locally expanding. In the case $A \subset \mathbb{S}_b$, A^* is an ω-limit set by Lemma 2.5, hence A^* is locally expanding (see Theorem 2.12 in [3]) and by Lemma 2.4, the set A is locally expanding as well. It remains to consider the case when $A = \mathbb{S}$. Since A is an ω-limit set and it has a non-empty interior, A is a cycle of intervals (see Theorem 1.1 in [6]). From this it follows that if $W \subset A$ is an interval, then W has a dense orbit in A and hence there is an $n \in \mathbb{N}$ such that $f^n(W) \cap W \neq \emptyset$. Therefore the union $\bigcup_{i=1}^{\infty} f^i(W)$ is dense in A and has finitely many component intervals. As this is true for every such interval W, it follows that A is locally expanding.

Assume that A is locally expanding. In the case $A \subset \mathbb{S}_b$ we can again prove the theorem by using our Lemmas 2.4 and 2.5, and Theorem 2.12 in [3]. It remains to consider the case when $A = \mathbb{S}$. By Lemma 2.7 the set A is a cycle of intervals and $f|_A$ is transitive. Thus the set A is an ω-limit set. □

Proof of Theorem 1.2. In the case when $X = I$ the theorem is proved in [3] (Theorem 3.1). It remains to consider the case when $X = \mathbb{S}$. We will prove this in several steps.

Case 1. Assume that $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} \subset \mathbb{S}_b$. Using our Lemma 2.5 and Theorem 3.1 in [3] we get that the set $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n^*}$ is an ω-limit set. By Lemma 2.5 the set $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n}$ is an ω-limit set as well.

Case 2. Next assume that $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} = \mathbb{S}$. Then it suffices to show that f is transitive. Take two non-empty open sets $V, W \subset \mathbb{S}$ and assume without loss of generality that they are disjoint.

Subcase 2.1. If there is an m such that ω_m intersects both V and W we are done since there are positive integers $p < q$ such that $f^p(x_m) \in V$ and $f^q(x_m) \in W$ and consequently, $f^{q-p}(V) \cap W \neq \emptyset$.

Subcase 2.2. If there is no such m, let $\{\omega_{n_i}\}_{i=1}^{\infty}$ be the subsequence of $\{\omega_n\}_{n=1}^{\infty}$ consisting of ω-limit sets intersecting V. Then $\omega_V = \bigcap_{k=1}^{\infty} \overline{\bigcup_{i=k}^{\infty} \omega_{n_i}} \subset \mathbb{S}_b$ for any $b \in W$, hence, according to the first part, $\omega_V = \omega_f(v)$ is an ω-limit set, and $a \in \omega_f(v)$ is its cluster point from the side T. Similarly, for some w, $\omega_f(w)$ is an ω-limit set intersecting W and such that a is its cluster point from the side T. Let $A = \omega_f(v) \cup \omega_f(w)$.

Subcase 2.2.1. If $A \neq \mathbb{S}$ then $A \subset \mathbb{S}_b$ for some b. We apply the result by Sharkovsky [7] which is also stated in [3]: If, for a map in $\mathcal{C}(I, I)$, two ω-limit sets have a common cluster point from the same side then their union is an ω-limit set. So, by Lemma 2.5 A is an ω-limit set since both $(\omega_V)^*$ and $(\omega_W)^*$ are and have a point a^* as a common cluster point from side T. We have the situation described in Subcase 2.1.

Subcase 2.2.2. $A = \omega_f(v) \cup \omega_f(w) = \mathbb{S}$. Since any ω-limit set in $\mathbb{S}$ is either nowhere dense or a finite union of non-degenerate intervals, and since $\omega_f(v) \cap W = \emptyset = \omega_f(w) \cap V$, both $\omega_f(v)$ and $\omega_f(w)$ are finite unions of intervals. If $\omega_f(v) \cap \omega_f(w)$ is infinite then the two ω-limit sets have an interval in common and the transitivity is easily proven. If the intersection $\omega_f(v) \cap \omega_f(w)$ would be finite then the condition with the T-side must be violated since the intersection contains a. □

Proof of Theorem 1.3. In the case when $X = I$ the theorem is proved in [3] (Theorem 3.2). It remains to consider the case when $X = \mathbb{S}$. Let $\{\omega_1, \omega_2, \ldots\}$ be a sequence of ω-limit sets converging in the Hausdorff metric to a set A. Choosing a subsequence (if necessary) we may also assume that there exists a point a, a side T of a and points $a_n \in \omega_n$, $a_n \neq a$ converging to a from T. As the original sequence converges to A, the subsequence does as well. To finish the proof it remains to use Theorem 1.2 and to show that $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} = A$. Since we consider Hausdorff metric and all the sets ω_n are closed then the set A is closed as well. Hence it is clear that $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} \supset A$. Consider the sequence of open sets $\{A_{1/n}\}_{n=1}^{\infty}$ where $A_\varepsilon := \{x \in X; \operatorname{dist}(x, A) < \varepsilon\}$,

$\mathrm{dist}(x, A) := \inf\{d(x, a); a \in A\}$ and d is the metric on X, and note that for every m there is a positive integer k such that $\overline{\bigcup_{n=k}^{\infty} \omega_n} \subset A_{1/m}$. Therefore $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} \subset A$. □

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